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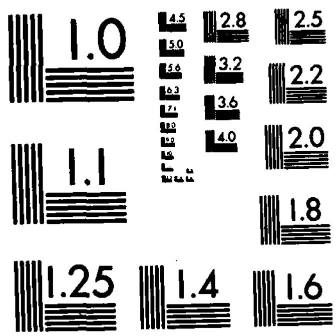

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RELATIVELY RECURSIVE RATIONAL CHOICE

by

Alain A. Lewis

Technical Report No.357
November 1981

Prepared Under
Contract ONR-N00014-79-C-0685, United States Office of Naval Research
Center for Research on Organizational Efficiency
Stanford University

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ABSTRACT

We have demonstrated previously (Lewis [1981]) that within the framework of recursive functions, a distinction must be made between representations of the paradigm of consumer choice, and realizations of a given representation. The present paper extends our previous framework to show, in brief fashion, that the concept of a recursive rational choice function defined as an effectively computable representation of Richter's [1971] concept of rational choice, attains by means of an application of Church's Thesis to the degrees of unsolvability associated with a classification of types of subsets of the natural numbers, a minimal bound in a measure of computational complexity entailed by its realization in an effectively computable manner.

RELATIVELY RECURSIVE RATIONAL CHOICE*

by

Alain A. Lewis**

I. Introduction

The purpose of the introduction is to acquaint the reader with the key concepts and terminology employed in the derivation of the principal result of our previous paper, to which the present work is to be considered as a sequel. The reader is encouraged to refer to the earlier paper (Lewis [1981]) for a more extensive exposition of the items that are summarized in the following discussion. The idea that is basic to the construction and definitions is that recursive structures of number theoretic character, by means of what is known as Church's Thesis†, represent a very general class of effectively computable procedures, that in another sense, typify ideal devices of artificial intelligence.

By a <u>recursive space of alternatives</u> is meant a pair $\langle R(\chi), \mathbb{F}_R \rangle$ where $R(\chi)$ is the image of a subset, $\chi \subsetneq \mathbb{R}^n_+$, that is compact and convex, in a <u>recursive metric space</u>, $M(\mathbb{R}^n)$, which is in turn comprised of n-tuples of <u>R-indices of recursive real numbers</u>; and where $\mathbb{F}_R$ is the class of all <u>recursive subsets of $R(\chi)$</u>. A <u>recursive choice</u> on $\langle R(\chi), \mathbb{F}_R \rangle$ is a set valued function defined on the class $\mathbb{F}_R$, $C : \mathbb{F}_R \to P(R(\chi))$,

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*This work was sponsored by Office of Naval Research Contract ONR-N00014-79-C-0685 at the Center for Research on Organizational Efficiency at Stanford University.

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†The discussion of Church's Thesis in Chapter 1 of Rogers [1967] is an excellent introduction to this concept.

such that for any $A \in \mathbb{F}_R$, $C(A) \subseteq A$. We call the choice on $<R(\chi), \mathbb{F}_R>$ a <u>recursive rational choice</u> if (1) there exists a binary relation $\geq : R(\chi) \times R(\chi) \to \{1,0\}$ termed the preference ordering; (2) there exists a function $f : R(\chi) \to \mathbb{N}$ such that if it should be true of α, $\beta \in R(\chi)$ that $\alpha \leq \beta$ then $f(\alpha) \geqq f(\beta)$; and (3) for any $A \in \mathbb{F}_R$, $C(A) = \{\alpha \in A : \forall \beta \in A \; f(\alpha) \geqq f(\beta)\}$. A recursive rational choice on $<R(\chi), \mathbb{F}_R>$ is thus rational in the sense of Richter [1971] and is representable also in Richter's sense. The <u>graph</u> of a choice on $<R(\chi), \mathbb{F}_R>$ is an enumerated collection of pairs of sets $<\mathbb{F}_{R_j}, C(\mathbb{F}_{R_j})>$ indexed by $j \in \mathbb{N}$ whose domain* is $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$, and whose co-domain is $\{C(\mathbb{F}_{R_j})\}_{j \in \mathbb{N}}$. The elements of the domain of graph (C) are members of the class $\mathbb{F}_R$, and it can be demonstrated† that if the choice on $<R(\chi), \mathbb{F}_R>$ is recursive rational, then the elements of the codomain $C(\mathbb{F}_{R_j})$ are also members of the class $\mathbb{F}_R$. A further result is that a <u>recursive rational choice</u> on $<R(\chi), \mathbb{F}_R>$ is also <u>recursively representable</u> in the sense that the components of graph (C) are all effectively computable by machine devices, in which case graph $(C) \subseteq \mathbb{F}_R \times \mathbb{F}_R$. A <u>recursive rational choice</u> on $<R(\chi), \mathbb{F}_R>$ is said to be <u>recursively realizable</u>, for suitable choice of domain, if and only if graph (C) is recursively solvable or equivalently if graph (C) is a recursive subset of $\mathbb{F}_R \times \mathbb{F}_R$. The

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*The domain is full if $\exists K \in \mathbb{N} \; \forall i \neq j > K \;\; \mathbb{F}_{R_j} \Delta \mathbb{F}_{R_i} \neq \phi$

†Cf. Lewis, [1981], Proposition V.2.

ment of the main theorem of Lewis ([1981] Theorem IV.4) which places a distinction between the notion of recursive representability and recursive realizability.

Theorem I.1: Allow $<R(\chi),\ \mathbb{F}_R>$ to be a recursive space of alternatives derived from the recursive metric space of $\mathbb{R}^n$, $M(\mathbb{R}^n)$, for $R(\chi)$ the recursive representation of a compact, convex subset of $\mathbb{R}^n_+$. Let $C : \mathbb{F}_R \to \mathbb{F}_R$ be the nontrivial recursive rational choice on $<R(\chi),\ \mathbb{F}_R>$ and select from the class of sequences $(\mathbb{F}_R)^{\mathbb{N}}$, element $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}} \subseteq \mathbb{F}_R$ that comprises a full domain for graph $(C) \subseteq \mathbb{F}_R \times \mathbb{F}_R$. Then, per fixed selection of $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}}$, graph (C) is recursively unsolvable and therefore is not recursively realizable.

The method of proof of the above theorem employs the fact that the restriction of graph (C) to its codomain was unsolvable by showing that any predicate that was adequate for describing the codomain belonged to a specific place in the Kleene-Mostowski Hierarchy that classifies subsets of the natural numbers by the complexity of their descriptions*. We presently exploit this feature of the proof to obtain a lower bound on the degree of difficulty of computing the graph of a recursive rational choice function by identifying the complexity of descriptive predicates in the hierarchy with a measure of computational

*A brief discussion of the Kleene-Mostowski Hierarchy is given in Appendix I of Lewis [1981]. A fuller discussion is in Rogers [1967] Ch.14, pp.301-334.

difficulty. The assertion of Church's Thesis which equates those functions that are effectively computable by ideal computing devices*, then becomes relative to the descriptive complexity of the predicates that are used to describe the function. Functions that are effectively computable in this sense of complexity are termed the relatively recursive functions.

II. Relatively Recursive Rational Choice

The concept of relative recursiveness originates with the work of the logician, Emil Post†, and is concerned with the reduction of a decision procedure for a given set of natural numbers, A, to that of another set of natural numbers, B. Intuitively speaking, a set A of natural numbers is reducible to a set B of natural numbers, if for a total predicate $\Psi : \mathbb{N} \to \{1, 0\}$, if the restriction to B, $\Psi_{|B}$, can be recursively realized, i.e., there exists a recursive $\Phi : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{cases} \Phi(n) = 1 \text{ when } \Psi_{|B}(n) \text{ is true} \\ \Phi(n) = 0 \text{ when } \Psi_{|B}(n) \text{ is false.} \end{cases}$$

Then there exists a recursive realization for the restriction of Ψ to A, $\Psi_{|A}$, i.e., a recursive $\Gamma(\Phi) : \mathbb{N} \to \mathbb{N}$ such that

*Cf. Lewis [1981], Ch.II, and Rogers [1967] p.130.

†"Degrees of Recursive Unsolvability", (Preliminary Report), Abstract Bulletin of A.M.S., Vol.54, [1948], pp.641-642.

$$\begin{cases} \Gamma(\Phi)(n) = 1 \text{ when } \Psi|_A \text{ is true} \\ \Gamma(\Phi)(n) = 0 \text{ when } \Psi|_A \text{ is false} \end{cases}$$

Alternatively, the set A is said to be recursive relative to B, or recursive in B.*

In a subsequent article, jointly authored with Kleene [1], Post develops more fully the concept of relative recursiveness in terms of degrees of unsolvability, which in turn is based on relations of reducibility between decision procedures for subsets of the natural numbers.† The subject of degrees of unsolvability has been developed into an extremely sophisticated branch of mathematical logic following the article by Post and Kleene. To attempt a discussion of the substance of the theory of degrees of unsolvability would exceed the bounds of the present paper, and we will confine ourselves to using only those items that bear on the relative recursiveness of rational choice, referring the interested reader to more comprehensive works.**

A possible means of interpreting the assertion of Theorem I.1 is, that when recursively represented on subsets of the natural numbers, a non-trivial rational choice function does not contain enough mathematical information in its graph to render the recursive solvability of its graph. This is not to say that there may be in fact other subsets of the natural numbers that do in fact contain enough mathematical

*The discussion in Rogers [1967], Ch.IX, pp.127-134.

†Cf. Ch. 13 of Rogers [1967].

**Gerald E. Sacks, Degrees of Unsolvability, Annals of Mathematics No. 55, Princeton University Press, [1955], and Joseph R. Shoenfield [1971].

information. We are then led to the natural inquiry of just how much mathematical information is required to recursively solve the graph of a recursive rational choice function. What we wish to show in the discussion that will follow is that by way of the notion of relative recursiveness, it is possible to characterize which subsets of the natural numbers contain enough information for the decision procedure of recursively representable rational choice. Alternatively phrased, we wish to inquire into its degrees of unsolvability.

We may first make the observation that there are three classic notions of reducibility of decision procedures for subsets of the natural numbers:

Definition I: A set A is generally recursive reducible to a set B if A is recursive in B*.

Definition II: A set A is Turing reducible to a set B if there exists a Turing machine that reduces the decision procedure for A to that of B. The reduction is performed by means of an oracle that provides requisite information about the set B in the computation of the set A as is required.†

Definition III: A set A is canonically reducible to a set B if both A in $\mathbb{N} - A$ are B-canonical sets in the sense of Post**.

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*This is due to Kleene, "Recursive Predicates and Quantifiers," Transactions A.M.S., Vol.53, [1945], pp.41-73.

†Cf. Rogers [1967], p.129.

**Post, [1944].

It is another significant feature of recursive function theory that the above three notions of reducility of decision procedures are all equivalent (cf. Post and Kleene [1954], p.379) and hence, no generality is lost in considering sets of natural numbers that are relatively recursive versus sets of natural numbers that are Turing reducible in discussing results on degrees of unsolvability.

Let us next denote the fact that a set A is recursive in a set B by the relation $A \leq_R B$. Then it can be shown* that for sets of natural numbers the following items are true:

(i) $\forall A[A \leq_R A]$

(ii) $\forall A \forall B \forall C[A \leq_R B \;.\!\wedge\!.\; B \leq_R C) \Rightarrow A \leq_R C]$

If we mean by $A \equiv_R B$ that both $A \leq_R B$ and $B \leq_B A$, from items (i) and (ii) we see that $\equiv_R$ is an equivalence relation among sets of natural numbers from which the following definition can be obtained.

<u>Definition IV</u>: The degree of unsolvability, with respect to $\leq R$, of a set A is defined to be:

$$[A] = \{B \subseteq \mathbb{N} : A \equiv_R B\} = dgA$$

In terms of the definition, the following facts concerning degrees of unsolvability can be derived.

*Cf. Rogers [1967] p.78 or Schoenfield [1971] p.44.

<u>Proposition II.1</u> : Allow A and B to be sets of natural numbers. Then

(i) $A \equiv_R B \Leftrightarrow dgA = dgB$.

(ii) If $a \leq b$ means that for some A and B $a = dgA$ and $b = dgB$, then $dgA \leq dgB \Leftrightarrow A \leq_R B$ and $dgA < dgB \Leftrightarrow (A \leq_R B \;.\wedge.\; B \nleq_R A)$.

We may view the concept of degrees of unsolvability in terms of a sort of relativized Church's Thesis, in that if $A \leq_R B$ maintains then we should think of A as at least as easy to, or alternatively not more difficult to compute by effective means as B is to compute by effective means. From this, $A \equiv_R B$ would mean that A and B are equally as difficult or equally as easy to compute by effective means. It can also be observed that the relation $\leq$ on the equivalence classes under $\equiv_R$ partially orders the set of degrees, and thus if the degree of a set is a measure of the difficulty involved in effectively computing that set, then the higher the degree under the order $\leq$, the more difficulty involved in effective computation.

If it should happen that set A is recursive, i.e., Σ_0-π_0 ,* then under the relation of relative recursiveness, for any set B, it happens to be true that $A \leq_R B$, and thus that $dgA \leq dgB$ for any dgB.

*The Σ_0-π_0 classification is the first classification in the Arithmetic Hierarchy. Cf. Lewis [1981] Appendix I or Rogers [1967] Ch.14.

Therefore, there is a smallest degree denoted as 0, and 0 is the degree of every recursive set A, i.e., every Σ_o-π_o set. On the other hand, if for some set A, it were true that dgA = 0, then the set is recursive, i.e., Σ_o-π_o . The latter assertion follows from the fact that for any recursive set B, $A \leq_R B$. Therefore we have the following :

<u>Proposition II.2</u> : A set of natural numbers A, is Σ_o-π_o if, and only if dgA = 0 .

From the proposition, and the relationship it provides between a recursive set of natural numbers and its degree of unsolvability we are led, by way of the structure of the partial order on the set of degrees, to certain deeper qualities of the Arithmetic Hierarchy by associating degrees of unsolvability with levels and compartments in the hierarchy of sets of natural numbers.

Let us now remark that Theorem I.1 can be given an interpretation at this point in terms of the theory of degrees that we have just developed. The gist of what Theorem I.1 says is that if we view a non-trivial recursive rational choice function as a correspondence between classes of recursive sets the graph of the correspondence cannot be recursive. By means of Theorem 4 Sec.5 p.24 of Schoenfield [1971], if G is the graph of a relation Γ, then dgG = dgΓ, and thus a further interpretation of Theorem I.1 is that, in viewing C as a correspondence, since graph (C) is not recursive and the degree of a recursive set is 0, dgC ≠ 0 by way of Proposition II.2 . In terms of a relativized Church's Thesis, if we say that dgC ≠ 0 then we merely say that the level of

difficulty incurred in an effectively computable realization of C is not as easy as that of the recursive sets. It is desirable however, to say more than this, to which purpose we now turn to the main result.

III. The Minimal Degree of Recursive Rational Choice

The purpose of this section is to provide a statement on the lower bound of the degree of unsolvability of recursive rational choice by means of associating degrees with the classification of sets provided by the Arithmetic Hierarchy. Observe first, that an alternative means of defining the components of the Arithmetic Hierarchy to that of using Kleene strings*, as was done in Lewis [1981], can be obtained by defining the Σ_n and π_n sets inductively as follows:

(i) The Σ_o-π_o sets are the recursive sets.

(ii) A set is Σ_{n+1} if it can be defined as:

$$\chi \in A \iff \exists y[(\chi,y) \in B]$$

for B, a π_n set.

(iii) A set is π_{n+1}

$$\chi \in A \iff \forall y[(\chi,y) \in B]$$

for B, a Σ_n set.

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*A Kleene string is a quantified expression in the first order predicate calculus of a recursive predicate on $\mathbb{N}$.

From this defining framework it is possible to prove the following propositions by means of induction on the arity of the set*, and give rules for when sets are Σ_n or π_n †.

<u>Proposition III.1</u>: If A is $\Sigma_n(\pi_n)$ and B is defined by $\chi \in B \iff \Psi(\chi) \in A$ where Ψ is a recursive function, then B is $\Sigma_n(\pi_n)$.

<u>Proposition III.2</u>: If A is Σ_m <u>or</u> π_m for some $m < n$, then A is Σ_n <u>and</u> π_n.

<u>Proposition III.3</u>: If A is $\Sigma_n(\pi_n)$, then $\mathbb{N} - A$ is $\pi_n(\Sigma_n)$.

<u>Proposition III.4</u>: If A and B are $\Sigma_n(\pi_n)$, then $A \cup B$ and $A \cap B$ are $\Sigma_n(\pi_n)$.

The first three propositions are in fact properties of the Arithmetic Hierarchy, and the last says that per fixed arity, the Σ_n and π_n sets are closed under set operations. These properties are in turn useful in enabling one to evaluate the degrees of the Σ_n and π_n sets.

<u>Lemma III.5</u>: A set of natural numbers A is Σ_{n+1} if and only if for some set B, $A \leq_{Re} B$** and B is π_n.

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*The arity of a Σ_n or π_n set is n.

†The proofs are found in Schoenfield [1971] pp.31-32.

**The relation $A \leq_{Re} B$ reads A is recursively enumerable in B and weakens $A \leq_R B$. Cf. Schoenfield [1971] p.24.

Proof: By definition, $A \leq_{Re} B$ is true when $\chi \in A \iff y[(\chi,y)] \in B]$. If A is Σ_{n+1} then by (ii) of the inductive definitions, $A \leq_{Re} B$ for $B\pi_n$.

Conversely, suppose that $A \leq_{Re} B$, and B is π_n and suppose I indexes A in B, then $\chi \in A \iff \chi \in T_I^B \iff \alpha[\alpha \subseteq B . \wedge . \chi \in T_I^{\alpha}]$, where T_I^B yields elements in accordance with I with an oracle for B, and α is a finite sequence of elements in B. One can visualize T_I^B as a Turing machine that lists the elements of A using information about the set B. To see that A is Σ_{n+1}, it will suffice to show that $\alpha \subset B$ and $\chi \in T_I^{\alpha}$ are Σ_{n+1}. By way of Proposition III.4, $\chi \in T_I^{\alpha}$ has a recursively enumerable graph and thus is Σ_1. By Proposition III.2 it is therefore Σ_{n+1}. We observe next that $\alpha \subset B \iff \forall n < \ln(\alpha)$* $[\Psi_{\alpha}(n) = \Psi_B(n)]$ where Ψ_{α} and Ψ_B are the definig predicates of α and B respectively. Since α is finite, by Proposition III.4, it will do to show that $\chi = \Psi_B(n)$ is Σ_{n+1}. However, the following expression verifies this by way of Proposition III.4 directly:

$$\chi = \Psi_B(n) \iff [(\Psi_B(n) = 1 . \wedge . n \in B) . \vee . (\Psi_B(n) = 0 . \wedge . n \in B)] \quad \text{Q.E.D.}$$

The next definition, when applied to the set of degrees, will provide us with the means to make an assertion about the relative difficult in effectively computing recursive rational choice.

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*$\ln(\alpha)$ is the length of α .

Definition V: For a set of natural numbers A, define the jump or completion of A as:

$$A' = \{\chi : \Phi_\chi^A(\chi) = 1\} = \{\chi : \chi \in T_\chi^A\}$$

where T_χ^A is the domain of Φ_χ^A, for Φ_χ^A a defining partial predicate for the set A with Gödel index χ.*

The completion of a set A always has the property that $A' \leq_{Re} A$, and that $A \leq_R A'$. Furthermore, if $A \leq_R B$ then $A' \leq_R B'$. From this latter fact when we consider the completion of the elements of the set of degrees, starting with the minimal degree 0, we can form an ascending chain: $0 \leq_R 0'$, $0' \leq_R 0''$, $0'' \leq_R 0'''$, $0''' \leq_R 0''''$, $0'''' \leq_R 0'''''$, ... This is significant in the light of the next lemma.

Lemma III.6: Let A be a Σ_n or a π_n set. Then $dgA \leq 0^n$, for 0^n the n^{th} completion of the recursive degree 0.

Proof: By induction, if we consider $n = 0$, if A is $\Sigma_o - \pi_o$, the result is trivial as $dgA = 0$ necessarily. Assume then that the Lemma is true for π_n sets, then from taking Lemma III.5 forward, and by the properties of the completion operator if A were Σ_{n+1} and B were π_n, $A \leq_R B$ implies that $dgA \leq dgB \leq 0^n$. The case for Σ_n sets can be obtained from Proposition III.3 by means of the same argument. Q.E.D.

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*Cf. Rogers [1967], p.132, p.135, and p.255.

†Cf. Rogers op cit Theorem I(a), p.255.

<u>Definition VI</u>: A set is said to be a <u>complete</u> Σ_n set (or <u>complete</u> π_n set) if A is Σ_n (or π_n), and if B is an arbitrary Σ_n set (or π_n set) $B \leq_R A$.

We now demonstrate the main result of the present paper.

<u>Theorem III.7</u> : Allow $C : \mathbb{F}_R \to \mathbb{F}_R$ to be a nontrivial *recursive* rational choice on a recursive space of alternatives $\langle R(\chi), \mathbb{F}_R \rangle$ derived from the recursive metric space of $\mathbb{R}^n$, $M(\mathbb{R}^n)$ for $R(\chi)$, the recursive representation of a compact, convex subset of $\mathbb{R}^n$. <u>Then for any fixed choice from $(\mathbb{F})^{\mathbb{N}}$ of full domain, the degree of unsolvability of graph (C) and therefore that of C, viewed as a correspondence on sequences in $(\mathbb{F}_R)^{\mathbb{N}}$ cannot be less than 0^2</u>.

<u>Proof</u> : We begin with the observation that the following concept of strong reducibility implies general recursive reducibility of Df. I.

<u>Definition VII</u>: Allow Φ_1 and Φ_2 to be elements of $\bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)^*$ such that Φ_1 is n-*ary*, and Φ_2 is m-*ary*. Then Φ_1 is strongly reducible to Φ_2 written $\Phi_1 << \Phi_2$ if there are partial recursive n-*ary* functions $f_1, \ldots, f_m$, for which $\Phi \subset \lambda^{\dagger}(\chi_1 \ldots, \chi_n)\ \Phi_2(f_1(\chi_1, \ldots, \chi_n), \ldots, f_m(\chi_1, \ldots, \chi_n))$.

We next observe that the following concept of strictness implies completeness as described in Df. VI.

<u>Definition VIII</u>: An element of $\bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$, Φ , is said to be strictly π_K (or Σ_K) if Φ is π_K (or Σ_K), and if any other $\Lambda \in \bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ is such that $\Lambda << \Phi$.

Then, from the fact that strong reducility implies general recursive reducibility if a relation is strictly π_K (or Σ_K), then its graph, in accordance with Df. VI is a complete π_K (or Σ_K) set. We have shown in Lewis [1981], Theorem IV.4, however, that the non-recursiveness of the graph of a recursive rational choice function per fixed choice of full domain, occurs in fact because its codomain is strictly Σ_2, and thus in light of the above equivalence, the codomain of graph (C) is a restriction of graph (C), i.e. codomain graph (C) $\subset$ graph (C) in the sense of Df.X of Lewis [1981], it follows that the degree of unsolvability for graph (C) cannot be less than the degree of unsolvability of the codoamin. For, allow graph (Γ) to be the codomain of graph (C). Then from Theorem 4 of section 5, p.24 of Schoenfield [1971], $dg(\Gamma) = dg(\text{graph } (\Gamma))$ and since $dg(\text{graph } (\Gamma)) = 0^2$ by Lemma III.6 if graph (Γ) is a complete Σ_2 set, then, from the fact that $\Gamma \subset C$, graph (C) = graph (Γ) $\cup$ graph ($C - \Gamma$) and by Proposition III.2 and Proposition III.4 applied in successsion,

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* $\bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ is the class of all relation on $\mathbb{N}$.

†This is Church's λ-notation, and $\lambda\chi\,\Phi(\chi)$ is read "the partial function $<\chi,y>$ that gives y as a value when χ takes an integer value.

$dg(\text{graph }(C)) \geqq 0^2$. The assertion for C viewed as a correspondence from $\mathbb{F}_R$ to $\mathbb{F}_R$ follows from another application of Theorem 4 of Schoenfield, by which $dg(\text{graph }(C)) = dgC$ and thus $dgC \geqq 0^2$. Q.E.D.

We have not, as of yet, verified a somewhat natural conjecture that the degree of unsolvability of a recursive representation for rational choice can be no more than 0^2. If true, by precisely bounding the degree of recursive rational choice in this fashion, further connections would be possible within the realm of contemporary theoretical computer science, as 0^2 happens to be the degree of unsolvability of the inherent ambiguity problem of computer science, and also coincides with the decision degree of finite classes*.

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*Cf. A. Ready and W. Savitch, "The Turing Degree of the Inherent Ambiguity Problem for Context-Free Languages", Theoretical Computer Science, Vol.1, [1976], pp.77-91, and Rogers [1967], pp.264-265.

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